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Marginal deformations with $\mathrm{U}(1)^3$ global symmetry

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Abstract: We generate new 11-dimensional supergravity solutions from deformations based on $\mathrm{U}(1)^3$ symmetries. The initial geometries are of the form $\mathrm{AdS}_4 \times Y_7$, where Y_7 is a 7-dimensional Sasaki-Einstein space. We consider a general family of cohomogeneity one Sasaki-Einstein spaces, as well as the recently-constructed cohomogeneity three $L^{p,q,r,s}$ spaces. For certain cases, such as when the Sasaki-Einstein space is $\mathbf{S}^7$, $Q^{1,1,1}$ or $M^{1,1,1}$, the deformed gravity solutions correspond to a marginal deformation of a known dual gauge theory.

Keywords: p-branes, M-Theory.

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1. Introduction

A marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory to $\mathcal{N} = 1$ theory preserving $\mathrm{U}(1)\times\mathrm{U}(1)$ global symmetry, along with a $\mathrm{U}(1)_R$ symmetry, provides a new type IIB supergravity background geometry [1] via the AdS/CFT correspondence [2 − 4]. The gravity dual of the original undeformed theory has an isometry group which includes $\mathrm{U}(1) \times \mathrm{U}(1)$. The marginal deformation of the gauge theory can be described by an $\mathrm{SL}(2, \mathbb{R})$ transformation acting on a 2-torus $\mathbf{T}^2$ in the gravity solution. This particular $\mathrm{SL}(2, \mathbb{R})$ transformation produces a non-singular geometry provided that the original geometry is non-singular. Gravity duals corresponding to marginal deformations of other field theories based on conifolds and toric manifolds are also given in [1].

A prescription for finding the marginal deformations of eleven-dimensional gravity solutions with $\mathrm{U}(1)^3$ global symmetry was also provided in [1], and was applied to the case of $\mathrm{AdS}_4 \times \mathbf{S}^7$. The isometry group of $\mathbf{S}^7$ is SO(8). The $\mathrm{U}(1)^3$ symmetries can be embedded in the SU(4) subgroup of SO(8), which implies that the deformed solution preserves two supersymmetries in three dimensions. With the appropriate coordinates, the angular directions corresponding to the three U(1)'s can be identified, and the metric can be written in a form that explicitly shows the three-torus $\mathbf{T}^3$ symmetry. An additional angle is related to the $\mathrm{SO}(2)_R = \mathrm{U}(1)_R$ R-symmetry, which is a symmetry of the usual 3-dimensional $\mathcal{N} = 2$ superconformal field theories. Two of the $\mathbf{T}^3$ angles are used to dimensionally reduce and T-dualize the solution to type IIB theory. Performing an $\mathrm{SL}(2, \mathbb{R})$ transformation and then T-dualizing and lifting back to eleven dimensions on the transformed directions yields a new 11-dimensional solution. This deformed solution has a warp factor, as well as an additional term in the 4-form field strength, which depends on the deformation parameter. In the limit of vanishing deformation parameter, the original $\mathrm{AdS}_4 \times \mathbf{S}^7$ solution is regained.

This prescription can be readily applied to other 11-dimensional solutions with geometries of the form $\mathrm{AdS}_4 \times Y_7$ provided that, in addition to the R-symmetry group, the isometry group contains $\mathrm{U}(1)^3$. Although the operator that we are adding to the lagrangian of the corresponding three-dimensional dual gauge theory is not known, we expect that this is a marginal deformation because the deformed theory is conformal. We will consider cases for which Y_7 is a 7-dimensional Sasaki-Einstein space. Then the initial 3-dimensional dual gauge has two supersymmetries and the deformation does not break any supersymmetry. A d-dimensional Sasaki-Einstein space can be defined as the Einstein base of a Calabi-Yau cone, and can always be written in canonical form as a U(1) bundle over an Einstein-Kähler metric. This U(1) corresponds to the R-symmetry of the gauge theory. The requirement that the global symmetry group of the gauge theory includes $\mathrm{U}(1)^3$ corresponds to the condition that the $\mathrm{U}(1)^3$ lies within the isometry group of the Einstein-Kähler base space. The isometry groups of most of the Sasaki-Einstein spaces we consider have SU(2) or SU(3) elements, which contain U(1) and $\mathrm{U}(1)^2$ subgroups, respectively. Thus, the resulting deformed spaces have isometry groups in which the SU(2) and SU(3) factors are replaced by U(1) and $\mathrm{U}(1)^2$ accordingly.

Until recently, few explicit metrics were known for Sasaki-Einstein spaces. A countably infinite number of 5-dimensional Sasaki-Einstein manifolds of topology $\mathbf{S}^2 \times \mathbf{S}^3$ has been constructed in [5, 6].[1] These $Y^{p,q}$ spaces are characterized by two coprime positive integers p and q. The marginal deformations of type IIB solutions with geometries of the form $\mathrm{AdS}_5 \times Y^{p,q}$ have already been considered in [1].

Higher-dimensional Sasaki-Einstein spaces were found in [7]. We will focus on the 7-dimensional spaces, which we refer to as $X^{p,q}$. These spaces are cohomogeneity one and include the previously-known homogeneous spaces $\mathbf{S}^7$, $Q^{1,1,1}$ and $M^{1,1,1}$ as special cases.[2] The 6-dimensional Einstein-Kähler base space of $X^{p,q}$ can be expressed as a 2-dimensional bundle over a 4-dimensional Einstein-Kähler space B_4. We have a couple of choices for the base space B_4, namely $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. For $B_4 = \mathbb{CP}^2$, $\mathbf{S}^7$ and $M^{1,1,1}$ arise as

[1]The $\mathcal{N} = 1$ superconformal gauge theories living on the worldvolume of the D3-branes, dual to 5-dimensional anti-de Sitter space times 5-dimensional Sasaki-Einstein manifolds, were constructed in [8]. See also the relevant works in [6, 9 – 32].

[2]A homogeneous space has an isometry group G which is transitive. In this case, the manifold can be expressed as a coset space G/Γ, where Γ is the isotropy group. On the other hand, when generic orbits have n dimensions less than the dimensionality of the space, then the space is said to be cohomogeneity n.

particular cases while, for $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$, $Q^{1,1,1}$ arises as a special case. The $X^{p,q}$ family of spaces can be further generalized for $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$ by rendering the characteristic radii of the two 2-spheres to be different [10, 11]. This is also a cohomogeneity one family of Sasaki-Einstein spaces. We will refer to these more general spaces as $Z^{p,q,r,s}$, which are characterized by four positive coprime integers p, q, r and s.

It has been found that the 5-dimensional $Y^{p,q}$ spaces can be generalized to cohomogeneity two spaces $L^{p,q,r}$ [33]. This was found by noting that the connection between Sasaki-Einstein spaces and special BPS scaling limit of euclideanized equal-angular-momenta Kerr-de Sitter black holes [9, 34] could be generalized to the case in which there are two different angular momenta. Likewise, the 7-dimensional $X^{p,q}$ with $B_4 = \mathbb{CP}^2$ can be generalized to cohomogeneity three spaces $L^{p,q,r,s}$, which is related to 7-dimensional Kerr-de Sitter black holes with three independent angular momenta.

This paper is organized as follows. In section 2, we find the marginal deformation of $\mathrm{AdS}_4 \times Q^{1,1,1}$. Next, we apply this deformation procedure to 11-dimensional geometries which include various 7-dimensional Sasaki-Einstein spaces. In section 3, we consider the infinite family of cohomogeneity one Sasaki-Einstein spaces $Z^{p,q,r,s}$, which include the spaces $\mathbf{S}^7$ and $Q^{1,1,1}$ as particular cases. We provide a brief overview of these spaces in subsection 3.1 and apply the deformations to them in subsection 3.2. In section 4, we consider the $M^{1,1,1}$ space (subsection 4.1) and its cohomogeneity three generalization $L^{p,q,r,s}$ (subsection 4.2), which also includes $\mathbf{S}^7$ as a particular case. A number of possible further directions are presented in section 5, including the application of this deformation procedure to other types of geometries. For the convenience of the readers, we have included a compendium of Sasaki-Einstein spaces in appendix A, which discusses how the various families of these spaces are related. Flow diagrams explicitly show how families of Sasaki-Einstein spaces are related in various limits, for both the 5 and 7-dimensional cases. We include some calculational details of the deformation of $\mathrm{AdS}_4 \times L^{p,q,r,s}$ in appendix B. Lastly, although the main theme of our paper is marginal deformations of 11-dimensional solutions, we have included the deformation of the type IIB solution $\mathrm{AdS}_5 \times L^{p,q,r}$ in appendix C, since the recently-obtained 5-dimensional $L^{p,q,r}$ spaces have not yet been discussed in this context.

2. Deforming $\mathrm{AdS}_4 \times Q^{1,1,1}$

The simplest examples of 7-dimensional Sasaki-Einstein spaces include the round $\mathbf{S}^7$, $Q^{1,1,1}$ and $M^{1,1,1}$. Since the case of $\mathbf{S}^7$ has already been done in [1], here we begin by considering $Q^{1,1,1}$. By putting a large number of N coincident M2-branes at a conical singularity and taking the near-horizon limit, we obtain the 11-dimensional solution [35 – 39] (See also [40 – 42])

$$ds_{11}^2 = \frac{1}{4} ds^2_{\mathrm{AdS}_4} + ds^2_{Q^{1,1,1}} \,, \qquad F_{(4)} = \frac{3}{8}\left(\frac{3}{8}\right)^{1/6} \omega_{AdS_4} \,, \tag{2.1}$$

where ω_{AdS_4} is the volume element of a unit AdS_4 spacetime. The four-form field strength has an extra $(3/8)^{1/6}$ factor compared with the case of the round 7-sphere, which arises from the ratio of the volumes of $Q^{1,1,1}$ and $\mathbf{S}^7$. The metric for $Q^{1,1,1}$ can be written as

$$ds^2_{Q^{1,1,1}} = \frac{1}{64}\left(d\psi + \sum_{i=1}^{3} c_{\theta_i} d\phi_i\right)^2 + \frac{1}{32}\sum_{i=1}^{3}\left(d\theta_i^2 + s_{\theta_i}^2 d\phi_i^2\right). \tag{2.2}$$

Topologically, $Q^{1,1,1}$ is a U(1) bundle over $\mathbf{S}_1^2 \times \mathbf{S}_2^2 \times \mathbf{S}_3^2$. The base is parameterized by the spherical coordinates (θ_i, ϕ_i) where $i = 1, 2, 3$ denotes the i^{th} 2-sphere[3] and the angle ψ parameterizes the U(1) Hopf fiber. The $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_3 \times \mathrm{U}(1)$ isometry group of $Q^{1,1,1}$ corresponds to the $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_3$ global symmetry and U(1) R-symmetry of the dual conformal field theory of [36].

We will now apply the deformation procedure of [1] to the above 11-dimensional supergravity solutions.[4] With regards to the required $\mathrm{U}(1)^3$ symmetry, one U(1) can be taken from each SU(2) of the isometry group. Therefore, this deformation preserves the U(1) R-symmetry. Since the original dual gauge theory is $\mathcal{N} = 2$ supersymmetric, the deformed gauge theory will also be $\mathcal{N} = 2$ supersymmetric. In other words, the three U(1)'s commute with the $\mathcal{N} = 2$ supercharge.

The 11-dimensional metric can be decomposed into a 3-dimensional piece, for which the $\mathbf{T}^3$ symmetry is explicit, and a remaining 8-dimensional piece, as given in [1, (A.1)]. In rewriting the above 11-dimensional metric in the form of (A.1), we take the "11^{th}" direction as the ϕ_3 direction. By reading off the $d\phi_3^2$ terms from (2.2) one gets

$$\Delta^{1/3} e^{4\phi/3} = \frac{1}{64}(c_{\theta_3}^2 + 2s_{\theta_3}^2).$$

The warp-factor Δ is the determinant of the 3×3 matrix represented by the metric for $D\phi_i$ and is given by

$$\Delta(\theta_1, \theta_2, \theta_3) = \frac{c_{\theta_3}^2 s_{\theta_1}^2 s_{\theta_2}^2 - \frac{1}{2}(-2 + c_{2\theta_1} + c_{2\theta_2}) s_{\theta_3}^2}{65536}.$$

From these two results, one can read off $e^{4\phi/3}$. From the expression of (2.2), one can decompose the 3-dimensional metric into a 2-dimensional metric for the subspace (ϕ_1, ϕ_2) and another term which takes the form

$$\Delta^{1/3} e^{4\phi/3} \left(D\phi_3 + N_1 D\phi_1 + N_2 D\phi_2\right)^2. \tag{2.3}$$

Now one can read off the term $d\phi_3 d\phi_1$ and the term $d\phi_3 d\phi_2$ which will determine N_1 and N_2 of [1]:

$$N_1 = -\frac{2c_{\theta_1} c_{\theta_3}}{(-3 + c_{2\theta_3})}, \qquad N_2 = -\frac{2c_{\theta_2} c_{\theta_3}}{(-3 + c_{2\theta_3})}.$$

Clearly, the 2-dimensional metric from the 11-dimensional point of view can be written as

$$\Delta^{1/3} e^{-2\phi/3} h_{mn} D\phi^m D\phi^n =$$

[3] We use the simple notation $c_{\theta_i} \equiv \cos\theta_i$ and $s_{\theta_i} \equiv \sin\theta_i$.

[4] We follow the notations from [1].

$$= \frac{(-2+c_{2\theta_1}+c_{2\theta_3})}{32(-3+c_{2\theta_3})}\left(D\phi_1 + \frac{2c_{\theta_1}c_{\theta_2}s^2_{\theta_3}}{-2+c_{2\theta_1}+c_{2\theta_3}}D\phi_2\right)^2 - \tag{2.4}$$
$$- \frac{c^2_{\theta_3}(-2+c_{2\theta_1}+c_{2\theta_3})s^2_{\theta_2}+(c^2_{\theta_2}(-3+c_{2\theta_3})s^2_{\theta_1}+2(-2+c_{2\theta_1}+c_{2\theta_3})s^2_{\theta_2})s^2_{\theta_3}}{16(-3+c_{2\theta_3})(-2+c_{2\theta_1}+c_{2\theta_3})}D\phi_2^2\,,$$

where we introduce the following notation $D\phi_i = d\phi_i + \mathcal{A}^i$ and the fields

$$\begin{aligned}
\mathcal{A}^1 &= \frac{8c_{\theta_1}s^2_{\theta_2}s^2_{\theta_3}}{H}d\psi\,,\\
\mathcal{A}^2 &= \frac{1}{\left[-\frac{1}{2}(-2+c_{c_{2\theta_1}}+c_{2\theta_2})\frac{1}{s^2_{\theta_1}c_{\theta_2}}+\frac{s^2_{\theta_2}c^2_{\theta_3}}{c_{\theta_2}s^2_{\theta_3}}\right]}d\psi\,,\\
\mathcal{A}^3 &= \frac{8c_{\theta_3}s^2_{\theta_1}s^2_{\theta_2}}{H}d\psi\,.
\end{aligned}$$

where

$$H = 5 - 3c_{2\theta_3} + c_{2\theta_1}(-3+c_{2\theta_2}+c_{2\theta_3}) + c_{2\theta_2}(-3+2c^2_{\theta_1}c_{2\theta_3})\,. \tag{2.5}$$

One can read off the metric h_{mn}, whose determinant is equal to 1, from (2.4). The fields ϕ^i, $\mathcal{A}^i$ and three-form gauge fields transform appropriately under the general $\mathrm{SL}(3,\mathbb{R})$ transformation of coordinates ϕ^i.

The remaining 8-dimensional metric is given by

$$\Delta^{-1/6}g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}ds^2_{AdS_4} + \frac{1}{32}\left(d\theta_1^2+d\theta_2^2+d\theta_3^2\right) + \frac{s^2_{\theta_1}s^2_{\theta_2}s^2_{\theta_3}}{4H}d\psi^2\,. \tag{2.6}$$

The 11-dimensional metric (2.1) can now be written in a form which makes the $\mathbf{T}^3$ more explicit by adding together (2.3), (2.4), and (2.6).

We now dimensionally reduce this solution along the ϕ_3 direction and T-dualize along the ϕ^1 direction. The resulting type IIB solution can be written in the form of in [1, (A.5) and (A.6)]:

$$\begin{aligned}
ds^2_{IIB} &= \frac{1}{h_{11}}\frac{d\phi_1^2}{\sqrt{\Delta}} + \frac{\sqrt{\Delta}}{h_{11}}D\phi_2^2 + e^{2\phi/3}g_{\mu\nu}dx^\mu dx^\nu\,,\\
B &= \left(\frac{2c_{\theta_1}c_{\theta_2}s^2_{\theta_3}}{-2+c_{2\theta_1}+c_{2\theta_3}}\right)d\phi_1\wedge D\phi_2 + d\phi_1\wedge\mathcal{A}^1\,,\\
e^{2\Phi} &= \frac{e^{2\phi}}{h_{11}}\,,\\
C^{(0)} &= -\frac{2c_{\theta_1}c_{\theta_3}}{(-3+c_{2\theta_3})}\,,\\
C^{(2)} &= \left(\frac{2c_{\theta_2}c_{\theta_3}s^2_{\theta_1}}{-2+c_{2\theta_1}+c_{2\theta_3}}\right)d\phi_1\wedge D\phi_2 - d\phi_1\wedge\mathcal{A}^3\,,\\
C^{(4)} &= -\left(\frac{3}{8}\right)^{7/6}(\omega_3\wedge d\phi_1 + *_8\omega_3\wedge d\phi_2)\,,
\end{aligned} \tag{2.7}$$

where $*_8$ is the Hodge dual relative to the 8-dimensional metric $g_{\mu\nu}$ and $d\omega_3 = \omega_{AdS_4}$.

We will now take a particular $\mathrm{SL}(2,\mathbb{R})$ transformation which will yield a regular deformed 11-dimensional solution, provided that the initial solution is regular [1]. The $\mathrm{SL}(2,\mathbb{R})$ transformation is parameterized by $\hat{\gamma} \equiv R^3\,\gamma$ in the $\phi_1 - \phi_2$ plane. The modified warp-factor and three-form [1] are given by

$$\Delta' = \frac{\Delta}{(1+\hat{\gamma}^2\Delta)^2} \equiv G^2\Delta, \qquad C'_{123} = -\frac{\hat{\gamma}\Delta}{1+\hat{\gamma}^2\Delta}\,,$$

with other quantities unchanged.

T-dualizing and lifting along the transformed directions results in the modified 11 dimensional solution

$$\begin{aligned}
ds^2_{11} &= G^{2/3}\Bigg[\frac{(-2+c_{2\theta_1}+c_{2\theta_3})}{32(-3+c_{2\theta_3})}\left(D\phi_1 + \frac{2c_{\theta_1}c_{\theta_2}s^2_{\theta_3}}{-2+c_{2\theta_1}+c_{2\theta_3}}D\phi_2\right)^2 - \\
&\quad - \frac{c^2_{\theta_3}(-2+c_{2\theta_1}+c_{2\theta_3})s^2_{\theta_2}+(c^2_{\theta_2}(-3+c_{2\theta_3})s^2_{\theta_1}+2(-2+c_{2\theta_1}+c_{2\theta_3})s^2_{\theta_2})s^2_{\theta_3}}{16(-3+c_{2\theta_3})(-2+c_{2\theta_1}+c_{2\theta_3})}D\phi_2^2 + \\
&\quad + \frac{(3-c_{2\theta_3})}{128}\left(D\phi_3 - \frac{2c_{\theta_1}c_{\theta_3}}{(-3+c_{2\theta_3})}D\phi_1 - \frac{2c_{\theta_2}c_{\theta_3}}{(-3+c_{2\theta_3})}D\phi_2\right)^2\Bigg] + \\
&\quad + G^{-1/3}\left[\frac{1}{4}ds^2_{\mathrm{AdS}_4} + \frac{1}{32}\left(d\theta_1^2+d\theta_2^2+d\theta_3^2\right) + \frac{s^2_{\theta_1}s^2_{\theta_2}s^2_{\theta_3}}{4H}d\psi^2\right], \qquad (2.8)\\
F_{(4)} &= \left(\frac{3}{8}\right)^{7/6}\left(\omega_{AdS_4} + \hat{\gamma}\sqrt{\frac{\Delta}{512H}}s_{\theta_1}s_{\theta_2}s_{\theta_3}\,d\theta_1 d\theta_2 d\theta_3 d\psi\right) - \hat{\gamma}\,d\left(\Delta G\,D\phi_1 D\phi_2 D\phi_3\right).
\end{aligned}$$

While the initial geometry was the direct product $\mathrm{AdS}_4 \times Q^{1,1,1}$, the above deformed geometry is a warped product of these spaces. The warp factor G depends on three of the internal directions of $Q^{1,1,1}$. Since $G^{-1} \geq 1$, the deformed geometry is guaranteed to be regular since the initial geometry was regular.

Later examples of 11-dimensional deformed solutions given in this paper will have a four-form field strength that has similar form to the case above. The main difference will be in the $d\theta_1 d\theta_2 d\theta_3 d\psi$ term, which arises from $*_8\,\omega_{AdS_4}$, where the Hodge dual is with respect to a particular 8-dimensional space $g_{\mu\nu}$.

So far, we have described a particular 7-dimensional space $Q^{1,1,1}$. This is a special case of the Einstein spaces $Q^{p,q,r}$, where p, q and r represent the winding numbers of the U(1) bundle over the three 2-spheres. The $Q^{p,q,r}$ space has an isometry $\mathrm{SU}(2)^3 \times \mathrm{U}(1)$, except $Q^{0,0,1}$. One can choose $\mathrm{U}(1)^3$ as a subgroup of $\mathrm{SU}(2)^3$. For the case of $Q^{0,0,1}$, the isometry group is given by $\mathrm{SU}(2)^4$. Therefore, it is also possible to consider three U(1)'s. It is straightforward to apply the above deformation procedure to all of these spaces. However, the $Q^{1,1,1}$ space which we considered above is the only case which is supersymmetric, with holonomy group SU(3).

3. Cohomogeneity one generalization of $Q^{1,1,1}$

3.1 The $Z^{p,q,r,s}$ spaces

We will now consider a class of cohomogeneity one Sasaki-Einstein spaces $Z^{p,q,r,s}$, which contain $Q^{1,1,1}$ and $\mathbf{S}^7$ as particular cases. A countably infinite number of 7-dimensional

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Sasaki-Einstein spaces were found in [7]. The corresponding metrics can be expressed as a circle bundle over a 6-dimensional Einstein-Kähler base space which, in turn, is a 2-dimensional bundle over a 4-dimensional Einstein-Kähler base space. For the case in which the 4-dimensional Einstein-Kähler space is a direct product of two equal-radius 2-spheres, this construction can be generalized such that the spheres have different characteristic radii ℓ_1 and ℓ_2 [10, 11]. The corresponding metric of these Sasaki-Einstein spaces Y_7 expressed in canonical form is

$$ds^2_{Y_7} = \left[d\psi' + 2A_{(1)}\right]^2 + ds^2_{EK_6}\,. \tag{3.1}$$

where the 6-dimensional Einstein-Kähler metric is

$$\begin{aligned} ds^2_{EK_6} = &\frac{1}{8}(\ell_1 - n_1\, y)(d\theta_1^2 + s_{\theta_1}^2 d\phi_1^2) + \frac{1}{8}(\ell_2 - n_2\, y)(d\theta_2^2 + s_{\theta_2}^2 d\phi_2^2) + \\ &+ \frac{1}{F(y)}dy^2 + \frac{F(y)}{64}(d\beta - n_1\, c_{\theta_1} d\phi_1 - n_2\, c_{\theta_2} d\phi_2)^2 \end{aligned} \tag{3.2}$$

and

$$A_{(1)} = \frac{1}{8}[\ell_1 c_{\theta_1} d\phi_1 + \ell_2 c_{\theta_2} d\phi_2 + y(d\beta - n_1\, c_{\theta_1} d\phi_1 - n_2\, c_{\theta_2} d\phi_2)]\,.$$

The Kähler form for $ds^2_{EK_6}$ is $J = dA_{(1)}$. The function $F(y)$ is given by

$$F(y) = \frac{\frac{4}{3}a + \frac{16\ell_1\ell_2}{n_1}(\ell_1 - 1)\, y - 8\delta\, y^2 + \frac{16}{3}(n_1\ell_2 - n_2 + 2n_2\ell_1)\, y^3 - 4n_1 n_2\, y^4}{(\ell_1 - n_1\, y)(\ell_2 - n_2\, y)}\,, \tag{3.3}$$

where $\delta \equiv \ell_2(2\ell_1 - 1) + \ell_1(\ell_1 - 1)n_2/n_1$. It is obvious that the above metric (3.1) has an isometry SU(2) $\times$ SU(2) $\times$ U(1) $\times$ U(1). One can obtain the above metric from the 7-dimensional Sasaki-Einstein metric in [11, (4)] by taking

$$(\ell_1, \ell_2, p, q) \rightarrow \frac{1}{8}(\ell_1, \ell_2, n_1, n_2), \qquad r \rightarrow -y, \qquad \tau' \rightarrow \frac{1}{8}\beta\,.$$

For vanishing a, the present $F(y)$ coincides with $c(r)^2$ in [11]. The value for the integration constant a in (3.1) reflects a choice in the origin of y. In [11], a was set to zero. Also, the metric has a rescaling symmetry under which

$$n_i \rightarrow \lambda\, n_i\,, \qquad a \rightarrow \frac{a}{\lambda^2}\,, \qquad y \rightarrow \frac{y}{\lambda}\,, \qquad \beta \rightarrow \lambda\,\beta\,. \tag{3.4}$$

This implies that only the ratio n_1/n_2 of the parameters n_1 and n_2 is nontrivial. It has been found that the metric (3.2) is an Einstein-Kähler solution provided that the algebraic constraint

$$\frac{n_1}{n_2} = \frac{\ell_1 - 1}{\ell_2 - 1} \tag{3.5}$$

is satisfied. The constraint (3.5) was not taken into account in the local expressions given in [10]. In particular, if we neglect this constraint by taking $\ell_i = 1$ and keeping n_1 and n_2 as independent paramaters, then our $F(y)$ given in (3.3) reduces to the expression given in [10].

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We have purposely kept repetitive parameters in order to more clearly see limiting cases. In the limit $\ell_1 = \ell_2 = 1$ and $n_1 = n_2 \equiv c$, the space Y_7 reduces to the seven-dimensional space found in [7]. In this case, a is now the nontrivial parameter. The limit enlarging one of the two U(1)'s into SU(2) symmetry

$$\ell_i \to 1\,, \qquad n_i \to 0\,, \qquad y \to c_{\theta_3}\,, \qquad a \to 6\,, \qquad \beta \to \phi_3\,,$$

gives rise to the metric of $Q^{1,1,1}$, the case discussed in previous section. On the other hand, in the limit

$$\ell_i \to 1\,, \qquad n_1 \to 1\,, \qquad n_2 \to -1\,, \qquad a \to 3\,, \tag{3.6}$$

the metric (3.2) reduces to the Fubini-Study metric on $\mathbb{CP}^3$, which is given by

$$\begin{aligned} ds_{\mathbb{CP}^3} &= d\mu^2 + \frac{1}{4}c_\mu^2\,(d\theta_1^2 + s_{\theta_1}^2 d\phi_1^2) + \frac{1}{4}s_\mu^2\,(d\theta_2^2 + s_{\theta_2}^2 d\phi_2^2) + \\ &\quad + \frac{1}{4}s_\mu^2\,c_\mu^2\,(d\beta - c_{\theta_1}d\phi_1 + c_{\theta_2}d\phi_2)^2\,, \end{aligned} \tag{3.7}$$

where we have used the coordinate transformation $y = -c_{2\mu}$. In this case, the Sasaki-Einstein metric (3.1) becomes that of a round $\mathbf{S}^7$, where

$$A_{(1)} = \frac{1}{8}(s_\mu^2 - c_\mu^2)\,d\beta + \frac{1}{4}c_\mu^2\,c_{\theta_1}d\phi_1 + \frac{1}{4}s_\mu^2\,c_{\theta_2}d\phi_2\,, \tag{3.8}$$

and $J = dA_{(1)}$ is the Kähler form for $\mathbb{CP}^3$.

We take y to lie in the range $y_1 \le y \le y_2$. Different coordinates have been used in order to find the regularity conditions for this metric [11]. In particular, regularity requires that

$$\frac{n_2 - n_1}{n_2} < \ell_1 < 1\,, \qquad 0 < \ell_2 < 1\,,$$

where we have chosen $0 \le n_1 \le n_2$. It has been found that the regular metrics are specified by two rational numbers, or by four integers p, q, r and s. Therefore, these spaces can be denoted by $Z^{p,q,r,s}$, in analogy with the 5-dimensional spaces $Y^{p,q}$ [12].

Consider the coordinate transformation

$$\psi' = \frac{1}{4}\psi\,, \qquad \beta = -\alpha + \psi\,.$$

We will now suppose that n_i is nonzero so that any such factors can be absorbed by the coordinates. One can then express the metric (3.1) as

$$\begin{aligned} ds_{Y_7}^2 &= \frac{1}{8}(\ell_1 - n_1\,y)(d\theta_1^2 + s_{\theta_1}^2 d\phi_1^2) + \frac{1}{8}(\ell_2 - n_2\,y)(d\theta_2^2 + s_{\theta_2}^2 d\phi_2^2) + \frac{1}{F(y)}dy^2 + \\ &\quad + \frac{F(y)}{16(F(y)+4y^2)}(d\psi + \ell_1 c_{\theta_1}d\phi_1 + \ell_2 c_{\theta_2}d\phi_2)^2 + \\ &\quad + \frac{F(y)+4y^2}{64}\Bigg[d\alpha + \left(1 - \frac{4y}{F(y)+4y^2}\right)d\psi + \\ &\qquad\qquad + \left(n_1 - \frac{4\ell_1\,y}{F(y)+4y^2}\right)c_{\theta_1}d\phi_1 + \left(n_2 - \frac{4\ell_2\,y}{F(y)+4y^2}\right)c_{\theta_2}d\phi_2\Bigg]^2. \end{aligned}$$

In the case of vanishing n_i, it is more appropriate to use the $Q^{1,1,1}$ metric given by (2.8).

3.2 Deforming $\mathbf{AdS}_4 \times Z^{p,q,r,s}$

The 7-dimensional Sasaki-Einstein space $Z^{p,q,r,s}$ can be used to construct a solution of 11-dimensional supergravity. This solution has the geometry $\mathrm{AdS}_4 \times Z^{p,q,r,s}$ and is given by

$$ds_{11}^2 = \frac{1}{4} ds^2_{AdS_4} + ds^2_{Y_7}\,, \qquad F_{(4)} = \frac{3}{8}\left(\frac{\mathrm{vol}\, Z^{p,q,r,s}}{\mathrm{vol}\, \mathbf{S}^7}\right)^{1/6} \omega_{AdS_4}\,.$$

The $\mathrm{U}(1)^3$ symmetry required for the deformation can be taken from the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ portion of the isometry group of $Z^{p,q,r,s}$, which still leaves a U(1) for the R-symmetry. This means that the $\mathcal{N} = 2$ supersymmetry of the dual gauge theory will be preserved by the deformation. The metric $ds^2_{Y_7}$ can be written in the following form which makes the $\mathrm{U}(1)^3$ symmetry explicit:

$$\begin{aligned} ds_{11}^2 =& \frac{1}{4} ds^2_{\mathrm{AdS}_4} + \frac{1}{8}(\ell_1 - n_1 y)\, d\theta_1^2 + \frac{1}{8}(\ell_2 - n_2 y)\, d\theta_2^2 + \frac{1}{F(y)} dy^2 + \\ &+ \frac{F(y)+4y^2}{64}\left[D\alpha + \left(n_1 - \frac{4\ell_1 y}{F(y)+4y^2}\right) c_{\theta_1}\, D\phi_1 + \left(n_2 - \frac{4\ell_2\, y}{F(y)+4y^2}\right) c_{\theta_2}\, D\phi_2\right]^2 + \\ &+ \frac{g}{8f} D\phi_1^2 + f\left[D\phi_2 + K f^{-1} \ell_1 \ell_2 c_{\theta_1} c_{\theta_2} D\phi_1\right]^2 + \frac{K}{8g}(\ell_1 - n_1 y)(\ell_2 - n_2 y) s^2_{\theta_1} s^2_{\theta_2}\, d\psi^2\,, \end{aligned} \tag{3.9}$$

where the coefficient functions are

$$\begin{aligned} K =& \frac{F(y)}{16(F(y)+4y^2)}\,, \qquad f = \frac{1}{8}(\ell_2 - n_2 y) s^2_{\theta_2} + K \ell_2^2 c^2_{\theta_2}\,, \\ g =& f(\ell_1 - n_1 y) s^2_{\theta_1} + K \ell_1^2 (\ell_2 - n_2 y) s^2_{\theta_2} c^2_{\theta_1} \end{aligned} \tag{3.10}$$

and the connections corresponding to ϕ^1, ϕ^2, and α are given by

$$\begin{aligned} \mathcal{A}^1 =& \frac{K}{g}\, \ell_1(\ell_2 - n_2 y) c_{\theta_1} s^2_{\theta_2}\, d\psi\,, \qquad \mathcal{A}^2 = \frac{K}{g}\, \ell_2(\ell_1 - n_1 y) c_{\theta_2} s^2_{\theta_1}\, d\psi\,, \\ \mathcal{A}^3 =& \left(1 - \frac{4y}{F(y)+4y^2}\right) d\psi - \left(n_1 - \frac{4\ell_1\, y}{F(y)+4y^2}\right) c_{\theta_1}\, \mathcal{A}^1 - \left(n_2 - \frac{4\ell_2\, y}{F(y)+4y^2}\right) c_{\theta_2}\, \mathcal{A}^2. \end{aligned}$$

We can read off the various ingredients of the metric (3.9) which we use to reduce and T-dualize the solution to type IIB theory:

$$\begin{aligned} \Delta^{1/3} e^{4\phi/3} =& \frac{F(y)+4y^2}{64}\,, \qquad \Delta = \frac{F}{8192 K}\left(g + 8K^2 \ell_1^2 \ell_2^2 c^2_{\theta_1} c^2_{\theta_2}\right), \\ N_1 =& \left(n_1 - \frac{4\ell_1\, y}{F(y)+4y^2}\right) c_{\theta_1}\,, \qquad N_2 = \left(n_2 - \frac{4\ell_2\, y}{F(y)+4y^2}\right) c_{\theta_2}\,. \end{aligned} \tag{3.11}$$

After applying an $\mathrm{SL}(2,\mathbb{R})$ transformation, we T-dualize and lift back up to eleven dimensions along the transformed coordinates. The resulting 11-dimensional deformed solution has the metric

$$\begin{aligned} ds_{11}^2 =& G^{-1/3}\left[\frac{1}{4} ds^2_{\mathrm{AdS}_4} + \frac{1}{8}(\ell_1 - n_1 y)\, d\theta_1^2 + \frac{1}{8}(\ell_2 - n_2 y)\, d\theta_2^2 + \frac{1}{F(y)} dy^2 + \right. \\ &\left. + \frac{K}{8g}\,(\ell_1 - n_1 y)(\ell_2 - n_2 y) s^2_{\theta_1} s^2_{\theta_2}\, d\psi^2\right] + \end{aligned}$$

$$+G^{2/3}\left[\frac{F(y)+4y^2}{64}\left[D\alpha+\left(n_1-\frac{4\ell_1 y}{F(y)+4y^2}\right)c_{\theta_1}D\phi_1+\left(n_2-\frac{4\ell_2 y}{F(y)+4y^2}\right)c_{\theta_2}D\phi_2\right]^2+\right.$$

$$\left.+\frac{g}{8f}\,D\phi_1^2+f\,[D\phi_2+Kf^{-1}\ell_1\ell_2 c_{\theta_1}c_{\theta_2}\,D\phi_1]^2\right], \tag{3.12}$$

where the coefficient functions K, f, and g are given by (3.10). While the initial geometry was the direct product $\mathrm{AdS}_4 \times Z^{p,q,r,s}$, the above deformed geometry is a warped product of these spaces with the factor given by

$$G^{-1} = 1 + \hat{\gamma}^2\,\Delta$$

and Δ is given by (3.11). The warp factor G depends on three of the internal directions of $Z^{p,q,r,s}$. Since $G^{-1} \geq 1$, the geometry (3.12) is regular since the initial geometry was regular.

Note that the deformation of $\mathrm{AdS}_4 \times Q^{1,1,1}$ discussed in the previous section cannot be obtained as a special case of (3.12). This is because the $Q^{1,1,1}$ metric (2.8) was expressed in different coordinates, which means that we performed the dimensional reductions along different directions. On the other hand, the deformation of $\mathrm{AdS}_4 \times \mathbf{S}^7$ arises from (3.12) in the limit (3.6).

4. $M^{1,1,1}$ and its cohomogeneity three generalization

4.1 Deforming $\mathbf{AdS}_4 \times M^{1,1,1}$

We will now consider the deformation of an 11-dimensional solution with the geometry [40, 42–47]

$$ds_{11}^2 = \frac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{M^{1,1,1}}\,, \qquad F_{(4)} = \frac{3}{8}\left(\frac{27}{128}\right)^{1/6}\omega_{AdS_4}\,,$$

where the homogeneous 7-dimensional Sasaki-Einstein space $M^{1,1,1}$ has the metric

$$ds^2_{M^{1,1,1}} = \frac{1}{256}\left[d\tau + 3\sin^2\mu\,(d\psi+\cos\theta\,d\phi) + 2\cos\tilde{\theta}\,d\tilde{\phi}\right]^2 + \frac{1}{32}(d\tilde{\theta}^2+\sin^2\tilde{\theta}\,d\tilde{\phi}^2) +$$

$$+\frac{3}{16}\left[d\mu^2+\frac{1}{4}\sin^2\mu\,\left(d\theta^2+\sin^2\theta\,d\phi^2+\cos^2\mu\,(d\psi+\cos\theta\,d\phi)^2\right)\right].$$

The corresponding $\mathcal{N}=2$ dual gauge theory in three-dimensions is known [36]. Topologically, $M^{1,1,1}$ space is a nontrivial U(1) bundle parameterized by τ over $\mathbb{CP}^2 \times \mathbf{S}^2$. The $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry group of $M^{1,1,1}$ corresponds to $\mathrm{SU}(3)\times\mathrm{SU}(2)$ global symmetry and $\mathrm{U}(1)_R$ R-symmetry of the dual conformal field theory [36].

By taking the two U(1)'s parameterized by ψ and ϕ from SU(3) group and one U(1) parameterized by $\tilde{\phi}$ from SU(2) group, we have a $\mathrm{U}(1)^3$ for the deformation while leaving the U(1) R-symmetry of the dual gauge theory untouched. Thus, the $\mathcal{N}=2$ supersymmetry will be preserved. We can rewrite the metric in a form which makes the $\mathrm{U}(1)^3$ symmetry explicit:

$$
\begin{aligned}
ds_{11}^2 &= \frac{1}{4}ds^2_{\mathrm{AdS}_4} + \frac{3}{16}\left(d\mu^2 + \frac{1}{4}s_\mu^2\, d\theta^2\right) + \frac{1}{32}d\tilde\theta^2 + \frac{1}{64}g\, c_\mu^2\, s_{\tilde\theta}^2\, d\tau^2 + \\
&\quad + \frac{3}{256f}s_\mu^2\left(D\psi + c_\theta\, D\phi + 2f c_{\tilde\theta} D\tilde\phi\right)^2 + \frac{3}{64}s_\mu^2\, s_\theta^2\, D\phi^2 + \frac{f}{32g}\, D\tilde\phi^2\,,
\end{aligned}
$$

where

$$
f^{-1} \equiv 4 - s_\mu^2\,, \qquad g^{-1} \equiv 2c_\mu^2\, c_{\tilde\theta}^2 + (4 - s_\mu^2)s_{\tilde\theta}^2
$$

and the gauge connections corresponding to $\phi, \tilde\phi$ and ψ are given by

$$
\mathcal{A}^1 = 0\,, \qquad \mathcal{A}^2 = g\, c_\mu^2\, c_{\tilde\theta}\, d\tau\,, \qquad \mathcal{A}^3 = f(1 - 2g\, c_\mu^2\, c_{\tilde\theta})^2\, d\tau\,.
$$

One can then read off the various ingredients which are used to reduce and T-dualize the solution to type IIB theory:

$$
\begin{aligned}
\Delta^{1/3}e^{4\phi/3} &= \frac{3}{256f}s_\mu^2\,, \qquad \Delta = \frac{9s_\mu^4 s_\theta^2}{2^{19}g}\,, \\
N_1 &= c_\theta\,, \qquad N_2 = 2f\, c_{\tilde\theta}\,.
\end{aligned}
$$

After applying an $\mathrm{SL}(2,\mathbb{R})$ transformation, we lift back up to eleven dimensions and find the deformed solution

$$
\begin{aligned}
ds_{11}^2 &= G^{-1/3}\Big[\frac{1}{4}ds^2_{\mathrm{AdS}_4} + \frac{3}{16}\left(d\mu^2 + \frac{1}{4}s_\mu^2\, d\theta^2\right) + \frac{1}{32}d\tilde\theta^2 + \frac{1}{64}g\, c_\mu^2\, s_{\tilde\theta}^2\, d\tau^2\Big] + \\
&\quad + G^{2/3}\Big[\frac{3s_\mu}{256f}\left(D\psi + c_\theta\, D\phi + 2f c_{\tilde\theta} D\tilde\phi\right)^2 + \frac{3s_\mu^2\, s_\theta^2}{64}\, D\phi^2 + \frac{f}{32g}\, D\tilde\phi^2\Big]\,, \qquad (4.1)
\end{aligned}
$$

where $G^{-1} = 1 + \hat\gamma^2\,\Delta \geq 1$ and this geometry is completely regular.

It is also straightforward to apply the above deformation procedure to the $M^{p,q,r}$ spaces, which have an $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry and $\mathrm{SO}(7)$ holonomy. In particular, $M^{0,1,0} = \mathbb{CP}^2 \times \mathbf{S}^3$ has an $\mathrm{SU}(3)\times\mathrm{SU}(2)^2$ isometry and $M(1,0) = \mathbf{S}^5 \times \mathbf{S}^2$ has an $\mathrm{SU}(4)\times\mathrm{SU}(2)$ isometry [48]. However, only the $M^{1,1,1}$ case is supersymmetric.

4.2 Deforming $\mathbf{AdS}_4 \times L^{p,q,r,s}$

We will now consider a class of cohomogeneity three Sasaki-Einstein spaces $L^{p,q,r,s}$, which include $M^{1,1,1}$ and $\mathbf{S}^7$ as special cases. The metric for $L^{p,q,r,s}$ is given by [33]

$$
\begin{aligned}
ds_{Y_7}^2 &= (d\tau + \sigma)^2 + \frac{Y(x)}{4xF(x)}dx^2 - \frac{x\,(1 - F(x))}{Y(x)}\left(\sum_{i=1}^{3}\frac{\mu_i^2}{\alpha_i}d\varphi_i\right)^2 + \\
&\quad + \sum_{i=1}^{3}\left(1 - \frac{x}{\alpha_i}\right)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) + \frac{x}{\left(\sum_{i=1}^{3}\frac{\mu_i^2}{\alpha_i}\right)}\left(\sum_{j=1}^{3}\frac{\mu_j}{\alpha_j}d\mu_j\right)^2 - \sigma^2\,, \qquad (4.2)
\end{aligned}
$$

where $\sigma, Y(x)$, and $F(x)$ are given by

$$\sigma = \sum_{i=1}^{3}\left(1-\frac{x}{\alpha_i}\right)\mu_i^2 d\varphi_i\,,$$

$$Y(x) = \sum_{i=1}^{3}\frac{\mu_i^2}{(\alpha_i - x)}, \qquad F(x) = 1-\left(\frac{\mu}{x}\right)\prod_{i=1}^{3}\frac{1}{(\alpha_i - x)}\,, \qquad \sum_{i=1}^{3}\mu_i^2 = 1.$$

This metric smoothly extends onto a complete and non-singular manifold if $pl_1 + ql_2 + \sum_{j=1}^{3} r_j \frac{\partial}{\partial\varphi_j} = 0$ for coprime integers (p, q, r_j). Note that $p + q = \sum_{j=1}^{3} r_j$.

We will now discuss the case in which the cohomogeneity one $X^{p,q}$ spaces of [7] are recovered. For equal $\alpha_i \equiv \alpha$, we can pull the $1/\alpha_i$ factors of $\left(1-\frac{x}{\alpha_i}\right)$ out of the summation symbol in (4.2). Next, one can write the 5-sphere metric as a Hopf fibration over $\mathbb{CP}^2$ [49]

$$\sum_{i=1}^{3}\left(d\mu_i^2 + \mu_i^2 d\varphi_i^2\right) = (d\psi + A)^2 + ds^2_{FS(2)}\,,$$

where $ds^2_{FS(2)}$ is the Fubini-Study metric on $\mathbb{CP}^2$ and A is a local potential for the Kähler form on $\mathbb{CP}^2$. Therefore, the three U(1)'s are enhanced to SU(3) $\times$ U(1). Moreover,

$$\sum_{i=1}^{3}\mu_i^2 d\varphi_i = (d\psi + A)\,.$$

Upon the coordinate transformation

$$x \rightarrow y + \frac{1}{3c}, \qquad \mu \rightarrow \frac{1-ac^2}{3c^4}, \qquad \alpha \rightarrow \frac{4}{3c}\,,$$

the second term in (4.2) becomes

$$\frac{(1-cy)^2}{(\frac{4a}{3} - 8y^2 + \frac{32}{3}cy^2 - 4c^2y^4)}dy^2 \equiv \frac{dy^2}{\widetilde{F}(y)}\,.$$

Here, $\widetilde{F}(y)$ is the same as the one in (3.3) when $\ell_1 = \ell_2$ and $n_1 = n_2 = c$. Then the metric can be written as

$$ds^2_{Y_7} = \left[d\tau + \frac{3}{4}(1-cy)(d\psi + A)\right]^2 + \frac{dy^2}{\widetilde{F}(y)} + \frac{9}{64}c^2\widetilde{F}(y)(d\psi + A)^2 + \frac{3}{4}(1-cy)\,ds^2_{FS(2)}\,, \tag{4.3}$$

which is a rescaled version of the cohomogeneity one generalization of $M^{1,1,1}$ given in [7]. The original U(1)^4 isometry of 7-dimensional Sasaki-Einstein space is enhanced to SU(3) $\times$ U(1)^2, where the SU(3) symmetry comes from $\mathbb{CP}^2$ and two U(1)'s are parameterized by above τ and ψ. The $c = 0$ limit reproduces $M^{1,1,1}$ space [7]. As observed in [7], the limit of $a = 1$ provides the space $\mathbb{CP}^3$ [50] corresponding to the last three terms of (4.3) with the replacement $U = c_\theta^2$ where the function U is the same as the one in [7] and the resulting 7-dimensional space is $\mathbf{S}^7$ (The structure of the first term of (4.3) gives the exact expression for U(1) bundle).

We will now turn to the deformation of the 11-dimensional solution given by

$$ds_{11}^2 = \frac{1}{4}ds_{\mathrm{AdS}_4}^2 + ds_{L^{p,q,r,s}}^2\,, \qquad F_{(4)} = \frac{3}{8}\left(\frac{\mathrm{vol}\; L^{p,q,r,s}}{\mathrm{vol}\;\mathbf{S}^7}\right)^{1/6}\omega_{AdS_4}\,,$$

where the volume of $L^{p,q,r,s}$ is given in [33]. The three U(1) symmetries which will be used for the deformation act by shifting the φ_1, φ_2, and φ_3 directions. The remaining U(1) symmetry corresponds to the R-symmetry of the dual gauge theory. As in all of the previous cases, the marginal deformation preserves the $\mathcal{N}=2$ supersymmetry.

Since the three-torus $\mathbf{T}^3$ has coordinates φ_i where $i=1,2,3$, it is straightforward to rewrite the above metric in terms of three-dimensional part and the eight-dimensional part. As we have done before, by looking at the $d\varphi_3^2$ from (4.2), we obtain

$$\Delta^{1/3}e^{4\phi/3} = \frac{\mu_3^2\left(Y(x)\alpha_3^2 - x\left[Y(x)\alpha_3 - (-1+F(x))\mu_3^2\right]\right)}{Y(x)\alpha_3^2}\,. \tag{4.4}$$

Now we have φ_3 dependent term as in (2.3) and the coefficient function N_1 and N_2 can be determined. The explicit form for them is given in the appendix B. By rearranging the metric, the two-dimensional metric can be written as and the coefficients K, L, and M are presented in the appendix B.

$$\Delta^{1/3}e^{-2\phi/3}h_{mn}D\varphi^m D\varphi^n = K\left(D\varphi_1 + LD\varphi_2\right)^2 + \left(\frac{M}{N}\right)D\varphi_2^2\,, \tag{4.5}$$

from which we find that

$$\Delta = \left(\frac{KM}{N}\right)^{3/2} e^{2\phi}\,,$$

which, together with (4.4), gives Δ. Finally, the eight-dimensional metric is written as

$$\begin{aligned}\Delta^{-1/6}g_{\mu\nu}dx^\mu dx^\nu &= \frac{1}{4}ds_{\mathrm{AdS}_4}^2 + \frac{Y(x)}{4xF(x)}dx^2 + \sum_{i=1}^{3}\left(1-\frac{x}{\alpha_i}\right)d\mu_i^2 + \\ &\quad + \frac{x}{\left(\sum_{i=1}^3 \frac{\mu_i^2}{\alpha_i}\right)}\left(\sum_{j=1}^{3}\frac{\mu_j}{\alpha_j}d\mu_j\right)^2 + Pd\tau^2\end{aligned} \tag{4.6}$$

by collecting the remaining $d\tau^2$ term and the explicit form for P is in the appendix B. After applying an SL$(2,\mathbb{R})$ transformation, we lift back up to eleven dimensions and find the deformed solution

$$ds_{11}^2 = G^{2/3}\Delta^{1/3}e^{-2\phi/3}h_{mn}D\varphi^m D\varphi^n + G^{-1/3}\Delta^{-1/6}g_{\mu\nu}dx^\mu dx^\nu\,, \tag{4.7}$$

where

$$G^{-1} = 1 + \hat{\gamma}^2\Delta\,.$$

This is a completely regular geometry.

Note that the deformation of $\mathrm{AdS}_4 \times M^{1,1,1}$ discussed in the previous subsection cannot be obtained as a special case of (4.7). This is because the $M^{1,1,1}$ metric (4.1) was expressed in different coordinates, which means that we performed the dimensional reductions along different directions. However, the deformation of $\mathrm{AdS}_4\times\mathbf{S}^7$ arises from (4.7) in the limit of equal α_i and $a=1$.

5. Further directions

We have generated deformations based on $U(1)^3$ symmetries of 11-dimensional geometries which involve various 7-dimensional Sasaki-Einstein spaces. The initial geometries have the form of a direct product of AdS_4 and a 7-dimensional Sasaki-Einstein space Y_7, and the deformed geometries are a warped product of these spaces. The warp factor depends on three of the internal directions of Y_7. In fact, supersymmetric AdS in warped spacetimes were previously obtained from direct products of AdS and an internal space [51]. Furthermore, the corresponding consistent Kaluza-Klein warped embeddings of gauged supergravities were found [52]. That is, the vacuum AdS solutions of these gauged supergravities give rise to warped products with internal spaces. However, in most of these cases, the warp factor (depends on only one of the internal directions) is singular. There are some cases which have a non-singular warp factor but then the internal space has orbifold-type conical singularities. The warped spacetimes discussed in [1] and in this paper are completely regular. Therefore, it would be interesting to see if one could construct consistent Kaluza-Klein warped embeddings of gauged supergravities whose AdS vacua give rise to some of these solutions. In particular, one might be able to deform previously-known sphere reductions whose initial vacuum geometries are of the form $AdS_n \times \mathbf{S}^m$.

The general class of Sasaki-Einstein spaces discussed in section 3 include $\mathbf{S}^7$ and $Q^{1,1,1}$ as special cases. The most interesting problem is what are the dual conformal field theories corresponding to M-theory on $AdS_4 \times Y_7$. For $Y_7 = Q^{1,1,1}$ space, it is known that the gauge theory is $SU(N)^3$ with three chiral superfields [36]. The field contents are represented by a quiver diagram where nodes represent the gauge groups and link matter fields in the bi-fundamental representation of the gauge group they connect. It would be interesting to construct the complete spectrum of 11-dimensional supergravity compactified on Y_7 and compare the Kaluza-Klein spectrum with that of the corresponding gauge theory in the large N limit.

Most probably, the classes of 7-dimensional Sasaki-Einstein spaces discussed in this paper do not exhaust all of the possibilities. For instance, we have noted that $\mathbf{S}^7$ arises as a special case of two separate branches of cohomogeneity one families of Sasaki-Einstein spaces $X^{p,q}$, one with a 4-dimensional base space $B_4 = \mathbb{CP}^2$ and the other with $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. The question arises as to whether these two branches can be encompassed by a more general family of Sasaki-Einstein spaces.

On a related note, the generalization of such spaces to larger cohomogeneity tends to involve breaking $SU(2)$ and $SU(3)$ elements of the isometry group into $U(1)$ and $U(1)^2$ subgroups, respectively. For example, the isometry group of $M^{1,1,1}$ is $SU(3)\times SU(2)\times U(1)$. The cohomogeneity one generalization $X^{p,q}$ with $B_4 = \mathbb{CP}^2$ involves replacing the $SU(2)$ with $U(1)$, and the further generalization to cohomogeneity three $L^{p,q,r,s}$ involves replacing the $SU(3)$ with $U(1)^2$. Thus, the fact that the isometry group of $Z^{p,q,r,s}$ contains two $SU(2)$ factors implies that there may be room for further generalization.

One can also apply the deformation procedure to cases for which do not involve a Sasaki-Einstein space. For example, one might consider geometries of the form $AdS_4 \times M^7$, where M^7 is a 3-sasakian or proper weak G_2 manifold. These are gravity duals of $\mathcal{N} = 3$

and $\mathcal{N} = 1$ superconformal field theories, respectively. An infinite family of M^7 spaces have been constructed in [53] as principle SO(3) bundles over Tod-Hitchin metrics. These spaces are parameterized by a single integer. A particular case of this family of spaces is $N^{0,1,0}$ [54 − 58]. The SU(3) × SU(2) isometry of $N^{0,1,0}$ contains $\mathrm{U}(1)^3$ as a subgroup, which can be used in the deformation procedure. The dual gauge theory has $\mathcal{N} = 3$ or $\mathcal{N} = 1$ supersymmetry, depending on the orientation of the space. We expect some of the supersymmetry to be broken by the deformation in the $\mathcal{N} = 3$ case but not in the $\mathcal{N} = 1$ case.

Another special case which is a proper weak G_2 manifold is the squashed 7-sphere [59, 60, 48], which we denote as $\widetilde{\mathbf{S}}^7$. This space can be described as a squashed SO(3) bundle over a 4-sphere which is entirely equivalent to the standard Fubini-Study metric [48]. In this scheme the squashing corresponds to changing the size of the SO(3) fiber relative to the $\mathbf{S}^4$ base space. The $\mathrm{AdS}_4 \times \widetilde{\mathbf{S}}^7$ solution of 11-dimensional supergravity is expected to be dual to a 3-dimensional $\mathcal{N} = 1$ gauge theory with SO(5) × SO(3) global symmetry [61] for the squashing with left-handed orientation. One can apply the techniques for generating marginal deformations by choosing two U(1)'s from SO(5) and one from SO(3).

One could also apply these deformations to gravity duals of field theories which exhibit RG flows away from a conformal fixed-point. This has already been done in [1] for a fractional D3-brane on a deformed conifold [62], which corresponds to a 4-dimensional $\mathcal{N} = 1$ gauge theory which flows from a UV conformal fixed-point to a confining theory in the IR region. Also, [32] has studied the deformations of a D5-brane wrapped on an $\mathbf{S}^2$ of a resolved conifold, which is another example of a gravity dual of an $\mathcal{N} = 1$ gauge theory [63]. There are also regular 11-dimensional solutions which exhibit flows from the geometries we have discussed. For example, the near-horizon region of certain deformed M2-branes have geometries which go from $\mathrm{AdS}_4 \times Q^{1,1,1}$ or $\mathrm{AdS}_4 \times M^{1,1,1}$ to other smooth geometries [64]. The 3-dimensional gauge theory interpretation of these solutions is that there is an RG flow from a UV conformal fixed-point to a confining gauge theory. Marginal deformations can be applied to these theories as well.

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A. Compendium of Sasaki-Einstein spaces

Until recently, the only explicitly known 5-dimensional Sasaki-Einstein metrics were the round metric on $\mathbf{S}^5$, the homogeneous metric on $T^{1,1}$ and their quotients. We now know of a countably infinite number of explicit cohomogeneity one 5-dimensional Sasaki-Einstein

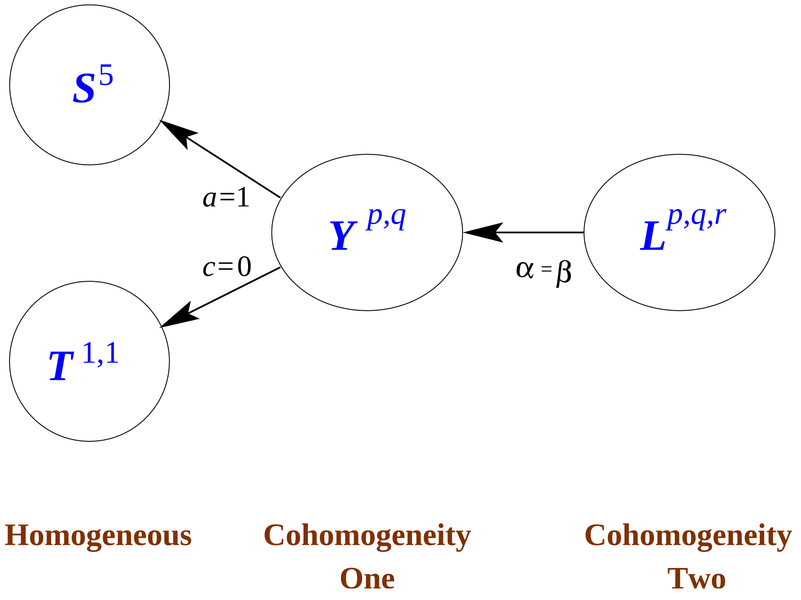

Figure 1: Flow diagram for 5-dimensional Sasaki-Einstein spaces.

metrics $Y^{p,q}$ [5, 6]. These spaces are specified by a single nontrivial parameter, though it is useful to keep the two parameters a and c in order to clearly see the special cases. In particular, $c = 0$ corresponds to the limit in which the space is $T^{1,1}$. In this case, a is a trivial rescaling parameter. On the other hand, the space is $\mathbf{S}^5$ for $a = 1$, in which case c is now a trivial rescaling parameter. The $Y^{p,q}$ spaces can be characterized by two relatively prime positive integers p and q. This class of spaces has been further generalized by taking the BPS limits of euclideanized Kerr-de Sitter black hole metrics with two independent angular momenta parameters α and β [33]. The resulting cohomogeneity two metrics $L^{p,q,r}$ are characterized by the positive coprime integers p, q and r. The $Y^{p,q}$ spaces lie in the subclass of the $L^{p,q,r}$ spaces for which $\alpha = \beta$. The flow diagram in figure 1 shows how all of the above 5-dimensional Sasaki-Einstein spaces are related. The $L^{p,q,r}$ metric is explicitly written in appendix C.

The cohomogeneity two $L^{p,q,r}$ spaces generally have $\mathrm{U}(1)^3$ isometry. For the subclass of spaces with cohomogeneity less than two, a $\mathrm{U}(1)$ or $\mathrm{U}(1)^2$ element of the isometry group gets enhanced to $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$, respectively. In particular, when $\alpha = \beta$, we get the cohomogeneity one $Y^{p,q}$ spaces with $\mathrm{SU}(2) \times \mathrm{U}(1)^2$ isometry. The further limit $c = 0$ yields the homogeneous space $T^{1,1}$, which has $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ isometry. Note that the remaining $\mathrm{U}(1)$ element corresponds to the R-symmetry of the dual field theory. The exception to the above is $\mathbf{S}^5$, whose isometry group is $\mathrm{SO}(6)$. In order to perform the type of deformation discussed in [1], it is crucial that the isometry group of all of these spaces contains $\mathrm{U}(1)^3$.

The higher-dimensional analog of the $Y^{p,q}$ spaces was found in [7]. Like the $Y^{p,q}$ spaces, these are also specified by a single nontrivial parameter which can be written in terms of two relatively prime positive integers p and q. Again, we will keep two parameters a and c in order to study special cases. We focus on the 7-dimensional spaces, which we refer to as

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$X^{p,q}$. Note that the 5-dimensional $Y^{p,q}$ spaces could be expressed as a U(1) bundle over a 4-dimensional Einstein-Kähler space which, in turn, was a 2-bundle over $\mathbf{S}^2$. Similarly, the 7-dimensional $X^{p,q}$ can be expressed as a U(1) bundle over a 6-dimensional Einstein-Kähler space which is itself a 2-bundle over a 4-dimensional Einstein-Kähler space B_4. We have a couple of choices for the base space B_4, namely $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We will first consider $B_4 = \mathbb{CP}^2$, which is quite analogous to the 5-dimensional case. $c = 0$ corresponds to the space $M^{1,1,1}$, for which a becomes the trivial rescaling parameter. $a = 1$ corresponds to the space $\mathbf{S}^7$, and now c is the trivial rescaling parameter. These $X^{p,q}$ spaces can be generalized to cohomogeneity three spaces $L^{p,q,r,s}$ [33], which can be found from the BPS limit of 7-dimensional euclideanized Kerr-de Sitter black holes which have three independent angular momenta α_i. The $X^{p,q}$ subclass of these spaces corresponds to all three angular momenta α_i being equal. The right-hand portion of figure 2 describes how these spaces are related. There is also an intermediate subclass which we have not included explicitly in figure 2. Namely, there is a subclass of the $L^{p,q,r,s}$ spaces with cohomogeneity two, for which only two of the α_i are equal.[5]

We now turn to the case of $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. This family of $X^{p,q}$ spaces also has a special $c = 0$ limit, in which case we have $Q^{1,1,1}$. However, $a = 1$ does not correspond to $\mathbf{S}^7$. In fact, we have not been able to identify this case with any previously-known Sasaki-Einstein space, though it does share a similarity with $\mathbf{S}^7$. In particular, like the Fubini-Study metric on $\mathbb{CP}^3$, the Einstein-Kähler base space is foliated by $T^{1,1}$. However, each $\mathbf{S}^2$ has an identical metric factor. In fact, this brand of $X^{p,q}$ can be further generalized by rendering the characteristic radii of the spheres to be different [10, 11]. This yields a more general cohomogeneity one family of Sasaki-Einstein spaces $Z^{p,q,r,s}$, which are characterized by four integers p, q, r, s. These spaces can be written in terms of four parameters ℓ_1, ℓ_2, n_1 and n_2, although only ℓ_1 and ℓ_2 are nontrivial parameters. This is because the n_i can be rescaled (3.4) such that only the ratio n_1/n_2 is significant. Also, in order for the $Z^{p,q,r,s}$ spaces to be Einstein, this ratio is written in terms of the ℓ_i (3.5). Nevertheless, it is useful to express the metric in terms of n_i, in order to see the special cases more clearly. In particular, when $\ell_i = 1$ and $n_1 = n_2$, we recover the $X^{p,q}$ spaces. On the other hand, when the $\ell_i = 1$ but $n_1 = -n_2$, we actually recover $\mathbf{S}^7$ by (3.7) and (3.8). Therefore, the $Z^{p,q,r,s}$ spaces are needed in order to encompass $Q^{1,1,1}$ (3.1) and $\mathbf{S}^7$ within a single family of cohomogeneity one Sasaki-Einstein spaces. The left-hand side of figure 2 describe the various special cases of the $Z^{p,q,r,s}$ family of spaces.

In general, the cohomogeneity three $L^{p,q,r,s}$ spaces have $\mathrm{U}(1)^4$ isometry. This is enhanced to $\mathrm{SU}(3) \times \mathrm{U}(1)^2$ isometry for the $X^{p,q}$ spaces with $B_4 = \mathbb{CP}^2$, and is further enhanced to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ for $M^{1,1,1}$. Since both the $Z^{p,q,r,s}$ spaces and $X^{p,q}$ with $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$ have cohomogeneity one, it is not surprising that they share $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ isometry. This is enhanced to $\mathrm{SU}(2)^3 \times \mathrm{U}(1)$ for the case of $Q^{1,1,1}$. The exception to the above is $\mathbf{S}^7$, which has SO(8) isometry. All of the above 7-dimensional spaces have isometry groups which contain $\mathrm{U}(1)^4$, which is necessary in order to perform deformations using the $\mathrm{U}(1)^3$ global symmetry while leaving the U(1) R-symmetry untouched.

[5] We thank Hong Lü for clarifying this point.

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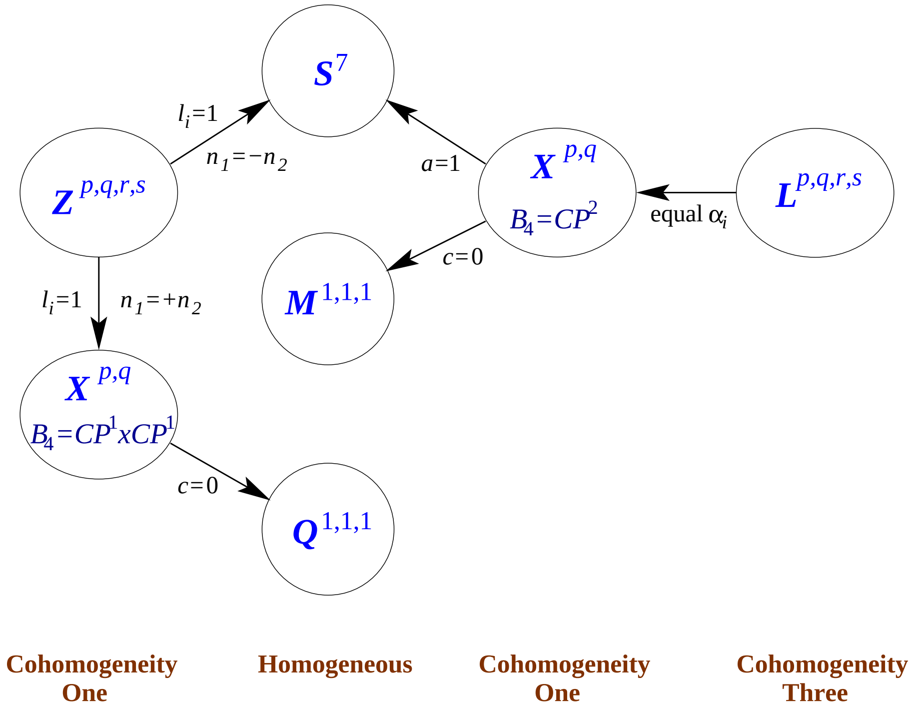

Figure 2: Flow diagram for 7-dimensional Sasaki-Einstein spaces.

B. The details for cohomogeneity three generalization of $M^{1,1,1}$

For convenience, we list all the coefficient functions N_1, N_2, K, L, M, N, P and gauge connections $\mathcal{A}^i$. The coefficient functions N_1 and N_2 are

$$N_1 = -\frac{(-1+F)\,x\alpha_3\mu_1^2}{\alpha_1\left(-Y\alpha_3^2 + x\left[Y\alpha_3 - (-1+F)\,\mu_3^2\right]\right)},$$
$$N_2 = -\frac{(-1+F)\,x\alpha_3\mu_2^2}{\alpha_2\left(-Y\alpha_3^2 + x\left[Y\alpha_3 - (-1+F)\,\mu_3^2\right]\right)}.$$

The coefficient functions K, L, M, and N appearing in (4.5) can be obtained as follows:

$$K = \frac{-\Delta^{1/3}e^{4\phi/3}N_1^2Y\alpha_1^2 + \mu_1^2\left(Y\alpha_1^2 - x\left[Y\alpha_1 - (-1+F)\,\mu_1^2\right]\right)}{Y\alpha_1^2},$$
$$L = \frac{\alpha_1\left[\Delta^{1/3}e^{4\phi/3}N_1N_2Y\alpha_1\alpha_2 - (-1+F)\,x\mu_1^2\mu_2^2\right]}{\alpha_2\left[\Delta^{1/3}e^{4\phi/3}N_1^2Y\alpha_1^2 + \mu_1^2\left(-Y\alpha_1^2 + x\left[Y\alpha_1 - (-1+F)\,\mu_1^2\right]\right)\right]},$$
$$\begin{aligned} M = {} & \Delta^{1/3}e^{4\phi/3}[N_2^2\alpha_2^2\mu_1^2\left(Y\alpha_1^2 - x\left[Y\alpha_1 - (-1+F)\,\mu_1^2\right]\right) - 2N_1N_2\,(-1+F)\,x\alpha_1\alpha_2\mu_1^2\mu_2^2 + \\ & + N_1^2\alpha_1^2\mu_2^2\left(Y\alpha_2^2 - x\left[Y\alpha_2 - (-1+F)\,\mu_2^2\right]\right)] + \\ & + \mu_1^2\mu_2^2(-Y\alpha_1^2\alpha_2^2 - x^2\left[Y\alpha_1\alpha_2 - (-1+F)\left(\alpha_2\mu_1^2 + \alpha_1\mu_2^2\right)\right] + \\ & + x\left[Y\alpha_1\alpha_2\,(\alpha_1+\alpha_2) - (-1+F)\left(\alpha_2^2\mu_1^2 + \alpha_1^2\mu_2^2\right)\right]), \end{aligned}$$
$$N = \alpha_2^2\left[\Delta^{1/3}e^{4\phi/3}N_1^2Y\alpha_1^2 + \mu_1^2\left(-Y\alpha_1^2 + x\left[Y\alpha_1 - (-1+F)\,\mu_1^2\right]\right)\right].$$

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The coefficient function P appearing in (4.6) is given by

$$
\begin{aligned}
P =\ & \frac{1}{KM\Delta^{1/3}e^{4\phi/3}\alpha_1^2\alpha_2^2\alpha_3^2} \times \\
& \times \Bigg(-M\Delta^{1/3}e^{4\phi/3}\alpha_2^2\left[\alpha_1\alpha_3\left(-\mu_1^2+N_1\mu_3^2\right)+x\left(\alpha_3\mu_1^2-N_1\alpha_1\mu_3^2\right)\right]^2+ \\
& \quad + K\Bigg(M\alpha_1^2\alpha_2^2\left[\Delta^{1/3}e^{4\phi/3}\alpha_3^2-\left(x-\alpha_3\right)^2\mu_3^4\right]- \\
& \qquad - \Delta^{1/3}e^{4\phi/3}\Bigg(L\alpha_2\left[\alpha_1\alpha_3\left(-\mu_1^2+N_1\mu_3^2\right)+x(\alpha_3\mu_1^2-N_1\alpha_1\mu_3^2)\right]+ \\
& \qquad\qquad + \alpha_1\left[\alpha_2\alpha_3(\mu_2^2-N_2\mu_3^2)+x(-\alpha_3\mu_2^2+N_2\alpha_2\mu_3^2)\right]\Bigg)^2\Bigg)\Bigg).
\end{aligned}
$$

The gauge connections are given by

$$
\begin{aligned}
\mathcal{A}^1 =\ & \frac{1}{KM}\Bigg[-\frac{\left(KL^2+M\right)\left(x-\alpha_1\right)\mu_1^2}{\alpha_1}+ \\
& + \frac{1}{\alpha_2\alpha_3}\Bigg(MN_1\alpha_2\left(x-\alpha_3\right)\mu_3^2+KL\times \\
& \quad \times\left[-\alpha_2\alpha_3\left(\mu_2^2+\left(LN_1-N_2\right)\mu_3^2\right)+x\left(\alpha_3\mu_2^2+\left(LN_1-N_2\right)\alpha_2\mu_3^2\right)\right]\Bigg)\Bigg], \\
\mathcal{A}^2 =\ & \frac{1}{M}\left[\left(1-\frac{x}{\alpha_2}\right)\mu_2^2+\frac{N_2\left(x-\alpha_3\right)\mu_3^2}{\alpha_3}+L\left(\left(-1+\frac{x}{\alpha_1}\right)\mu_1^2+\frac{N_1\left(-x+\alpha_3\right)\mu_3^2}{\alpha_3}\right)\right], \\
\mathcal{A}^3 =\ & -\frac{1}{\Delta^{1/3}e^{4\phi/3}}\Bigg(-\mu_3^2+\frac{x\mu_3^2}{\alpha_3}+\frac{\Delta^{1/3}e^{4\phi/3}N_2}{M}\times \\
& \times\Bigg[\left(1-\frac{x}{\alpha_2}\right)\mu_2^2+\frac{N_2\left(x-\alpha_3\right)\mu_3^2}{\alpha_3}+ \\
& + L\left(\left(-1+\frac{x}{\alpha_1}\right)\mu_1^2+\frac{N_1\left(-x+\alpha_3\right)\mu_3^2}{\alpha_3}\right)\Bigg]+ \\
& + \frac{1}{KM}\Bigg[\Delta^{1/3}e^{4\phi/3}N_1\times \\
& \times\Bigg(-\frac{(KL^2+M)(x-\alpha_1)\mu_1^2}{\alpha_1}+\frac{1}{\alpha_2\alpha_3}\times \\
& \times\Bigg[MN_1\alpha_2\left(x-\alpha_3\right)\mu_3^2+ \\
& + KL\Bigg(-\alpha_2\alpha_3\left(\mu_2^2+\left(LN_1-N_2\right)\mu_3^2\right)+ \\
& + x\left[\alpha_3\mu_2^2+\left(LN_1-N_2\right)\alpha_2\mu_3^2\right]\Bigg)\Bigg]\Bigg)\Bigg]\Bigg).
\end{aligned}
$$

C. Deforming $\mathbf{AdS}_5 \times L^{p,q,r}$

We have focused on deformations of supergravity solutions with $\mathrm{U}(1)^3$ global symmetry. Here we note that an analogous procedure can also be applied to the cohomogeneity two 5-dimensional Sasaki-Einstein space $L^{p,q,r}$ found in [33]. A solution of type IIB supergravity is given by

$$ds_{10}^2 = ds_{\mathrm{AdS}_5}^2 + ds_{L^{p,q,r}}^2\,, \qquad F_{(5)} = \frac{16\pi N}{\mathcal{V}}\left(\omega_{\mathrm{AdS}_5} + *\omega_{\mathrm{AdS}_5}\right),$$

where N is the number of D3-branes and $\mathcal{V} = \mathrm{vol}(L^{p,q,r})/\mathrm{vol}(\mathbf{S}^5)$, which is generically irrational and less than one. The volume of $L^{p,q,r}$ is given in [33]. The five-dimensional Sasaki-Einstein space $L^{p,q,r}$ has an isometry $\mathrm{U}(1)^3$ and is given by

$$\begin{aligned} ds_{L^{p,q,r}}^2 &= (d\tau+\sigma)^2 + \frac{\rho^2}{4\Delta_x}\,dx^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + \frac{\Delta_x}{\rho^2}\left(\frac{s_\theta^2}{\alpha}d\phi + \frac{c_\theta^2}{\beta}d\psi\right)^2 + \\ &\quad + \frac{\Delta_\theta s_\theta^2\,c_\theta^2}{\rho^2}\left[\left(1-\frac{x}{\alpha}\right)d\phi - \left(1-\frac{x}{\beta}\right)d\psi\right]^2, \end{aligned}$$

where

$$\begin{aligned} \sigma &= \left(1-\frac{x}{\alpha}\right)s_\theta^2\,d\phi + \left(1-\frac{x}{\beta}\right)c_\theta^2\,d\psi\,, \\ \Delta_x &= x(\alpha-x)(\beta-x)-1\,, \qquad \rho^2 = \Delta_\theta - x\,, \\ \Delta_\theta &= \alpha\,c_\theta^2 + \beta\,s_\theta^2\,. \end{aligned}$$

We put $\mu = 1$ in the last term of Δ_x. When $p = q = r = 1$(according to [33], $\alpha = \beta$), implying $\mathrm{SU}(2)^2\times\mathrm{U}(1)$ symmetry, the metric becomes $T^{1,1}$ space and When $\mu = 0$, one obtains round 5-sphere.

The two $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry is associated with shifts in ϕ and ψ. In order to make the $\mathbf{T}^2$ more explicit, we express the metric as

$$\begin{aligned} ds_{10}^2 &= ds_{\mathrm{AdS}_5}^2 + \frac{\rho^2}{4\Delta_x}\,dx^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + N\,d\tau^2 + \frac{c_\theta^2}{\beta^2\rho^2 L}\,(d\psi + \beta\rho^2 LM\,d\tau)^2 + \\ &\quad + \frac{s_\theta^2}{\alpha^2\rho^2 K}\left(d\phi - \frac{\alpha\rho}{\beta}c_\theta^2\,K\,d\psi + \alpha\rho^2(\alpha-x)K\,d\tau\right)^2, \end{aligned}$$

where the coefficient functions are

$$\begin{aligned} K^{-1} &= (\alpha-x)^2(\Delta_\theta - x\,s_\theta^2) + \Delta_x s_\theta^2\,, \\ L^{-1} &= (\beta-x)^2(\Delta_\theta - x\,c_\theta^2) + \Delta_x c_\theta^2 - \rho^2 K\,s_\theta^2\,c_\theta^2\,, \\ M &= (\beta-x) + (\alpha-x)\rho K\,s_\theta^2\,, \\ N &= 1 - (\alpha-x)^2\rho^2 K\,s_\theta^2 - \rho^2 LM^2\,c_\theta^2\,. \end{aligned}$$

The deformed metric is given by

$$\begin{aligned} ds_{10}^2 &= G^{-1/4}\left[ds_{\mathrm{AdS}_5}^2 + \frac{\rho^2}{4\Delta_x}\,dx^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + N\,d\tau^2\right] + \\ &\quad + G^{3/4}\left[\frac{c_\theta^2}{\beta^2\rho^2 L}(d\psi + \beta\rho^2 LM\,d\tau)^2 + \right. \\ &\qquad\qquad \left. + \frac{s_\theta^2}{\alpha^2\rho^2 K}\left(d\phi - \frac{\alpha\rho}{\beta}c_\theta^2\,K\,d\psi + \alpha\rho^2(\alpha-x)K\,d\tau\right)^2\right], \end{aligned}$$

where

$$G^{-1} = 1 + \hat{\gamma}^2\frac{s_\theta^2 c_\theta^2}{\alpha^2\beta^2\rho^4 LK}\,.$$

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